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Application of theorem of minimum potential energy to a complex structure Part I: two-dimensional analysis

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Abstract

A non-circular shell cross-section with flat sides and circular arc corners is analyzed using the theorem of minimum potential energy. Two-dimensional, plane strain assumptions are utilized, and the potential energy (PE) expression for the structure is developed, including first-order transverse shear deformation effects. The unknown displacements are represented by power series, and the PE expression is rewritten in terms of the summation convention for the power series. The variation of the PE expression is taken, leading to a linear system of equations that is solved for the unknown power series coefficients. With the displacements determined, stresses are calculated for a composite sandwich construction. Excellent agreement is found with other analytical methods and with finite element analyses.

Keywords: Non-circular shell; Minimum potential energy; Variational; Power series; Finite elements

1. Introduction

The ultimate goal of the current analysis method is the investigation of three-dimensional cylindrical shells with non-circular cross-sections subjected to constant internal pressure (see Figs. 1 and 2). The term "non-circular" is very broad, and could cover anything from regular shapes such as ovals, ellipses, triangles or squares to more unusual shapes such as multi-lobed, "wavy" (i.e., a sinusoidal variation about and around the mean perimeter) or irregular. Such shapes could be either open or closed, with or without symmetry. In the current research a closed, symmetric cross-section with a "rounded square" profile is examined. This profile consists of straight, flat sections at the top, bottom, and sides, connected by circular arc corners. The two-dimensional case is analyzed first as a stepping stone to the full three-dimensional analysis. A more detailed treatment of the subject is found in Preissner (2002).

The theorem of minimum potential energy (MPE) is used to bridge the gap between general, closed-form analytical solutions and extensive finite element analyses. In the current case, the complicated structure makes a closed-form solution impractical. The desire for eventual numerous and rapid trade studies weighs

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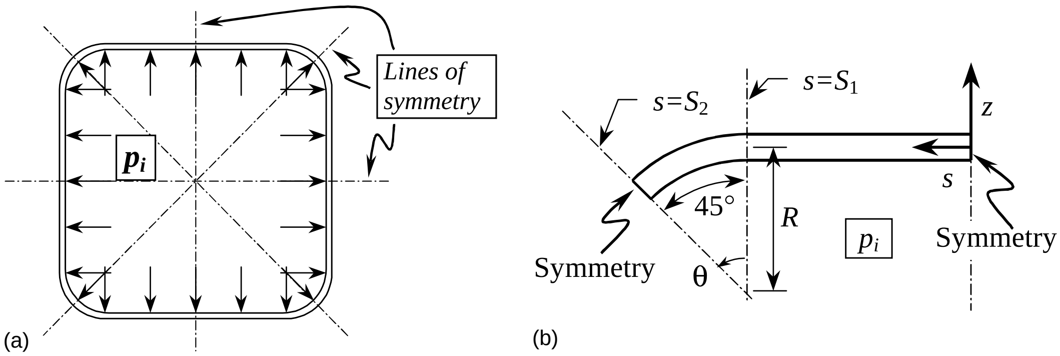

Fig. 1. (a) Full non-circular cross-section shape. (b) Details of the geometry.

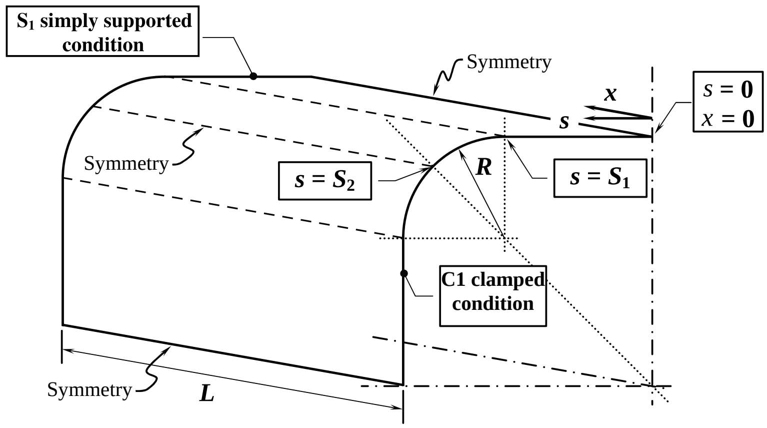

Fig. 2. Representation of shell mid-plane for three-dimensional problem.

against finite elements. The MPE method is used in the current case to find practical solutions to complicated problems in a reasonable form. The structural shape examined has not been extensively investigated, and offers potential benefits in real-world applications such as trucking and aircraft. The application of the MPE method as described is also unique.

2. Previous research on application of MPE to shell structures

The prime motivation behind the use of MPE methods is to find approximate solutions to difficult problems that cannot be solved in closed form. In many instances, this provides an alternative to a full three-dimensional elasticity solution or extensive finite element analyses. There are two major divisions of efforts: (1) using MPE methods to obtain governing equations and then find solutions through approximations of the displacements or stresses (e.g., Wenda, 1985; Tamurov, 1990; McDaniel and Ginsberg, 1993; Wu and Liu, 1994; Soldatos and Messina, 1998) or (2) using MPE to develop advanced finite element formulations of complicated problems (e.g., Crull and Basu, 1994; Chroscielewski et al., 1997).

Due to the advanced nature of the formulations, nearly all work examined included the effect of transverse shear, either linear (He, 1992), second order (e.g., Tamurov, 1990; Sklepus, 1996), or third order (e.g., Wu and Liu, 1994; Wu et al., 1995) in the thickness variable. By examining zeroth-, first-, and third-order shear effects, Messina and Soldatos (1999) show that agreement with three-dimensional elasticity solutions improves as the order increases. Wu and associates also include the effect of transverse normal stress and obtain excellent agreement (four to five significant figures) to prior elasticity solutions.

While Ying-guang et al. (1992) and Crull and Basu (1994) use a more complicated Reissner mixed formulation, whereby they assume both stresses and displacements, most authors assume only displacements. McDaniel and Ginsberg (1993) observe that "much physical insight is gained by viewing displacement expansions as basis functions for a variational solution...". A number of different types of expansions are used for the displacements, including trigonometric series (Wenda, 1985; Wu and Liu, 1994; Xiaoyu, 1997), Legendre polynomials (Crull and Basu, 1994), Bezier functions (Kumar and Singh, 1997), and polynomials of one of the coordinate parameters (Soldatos and Messina, 1998). The best results are obtained when different forms are used in the various coordinate directions, based on the boundary conditions of the edges (Wu and Liu, 1994; Messina and Soldatos, 1999). Methods for solving the subsequent systems of equations become complicated.

Importantly, Soldatos and Messina (1998) make four observations: (1) if "the set of basis functions is complete in the space of the functions that satisfy the edge boundary conditions assumed, the main task that remains is then to determine the unknown constant coefficients of such an infinite series" which is achieved through a variational (Ritz) or error minimization (Galerkin) approach, (2) sometimes a series approach is considered "approximate", but if enough terms can be taken, the solution will converge to the exact one, (3) "an inappropriate basis may yield a very slowly converging series solution or to [sic] cause severe numerical instabilities", and (4) an orthonormal basis of functions performs better than just an orthogonal basis by improving numerical stability by keeping matrix condition numbers lower.

It is notable that most of the analyses cover vibrations of laminated shells, as this is a difficult problem not easily solvable in closed form or economically by finite elements. The analyses covering stresses and displacements usually employ more complicated theories, such as layerwise formulations that include both transverse shear and transverse normal stress (Wu and Liu, 1994). However, despite (or perhaps because of) such complications, nearly all analyses consider what this author would term "simple" structures. In other words, continuous singly- or doubly-curved shells, possibly of unusual planform. Not truly "simple", yet the point is that they are regular, continuous structures, not a combination of different pieces. Only two papers cover non-continuous structures (a shell with edges restricted by beams, Wenda (1985), and the exhaustive development of finite element methods for complex shells of Chroscielewski et al. (1997)). These two formulations were not applicable to the current efforts.

In conclusion, it is seen that energy methods are good for situations not readily solvable by other means, e.g., the combined structure with beams on edges (Wenda, 1985), the vibration of turbine blades (Lim et al., 1998), or the vibration of open shells (Messina and Soldatos, 1999). None of the analyses address structures as complicated as the current research. These energy methods fill a solution niche for problems too complicated for closed form, and too expensive or expansive for current finite element methods.

3. Previous research on non-circular cylindrical shells

The general subject of shell analysis has been a fertile area for research for more that a century (Love, 1888). Over this period, there have been more books and papers written on the subject than can be listed here. Some significant texts include Flügge (1943), Timoshenko and Woinowsky-Krieger (1959), Dym (1974), and Vinson (1993), among many others. Surveys of the field include papers by Naghdi (1956), Singer (1982), Simitses (1986), Kapania (1989), and Simitses (1996). The subject of shell theory is very broad and very deep, and a complete review of shell theory is beyond the scope of this effort.

A detailed review of previous research in this specific area has been given in Preissner (2002). In this review, it was seen that the large majority of the research has been performed on the vibratory behavior of cross-section shapes that could be described in so-called "closed form" (e.g., Bergman, 1973; Suzuki et al., 1983; Soldatos, 1984; Suzuki and Leissa, 1990; Kumar and Singh, 1993; Suzuki et al., 1996). That is, the shape is an oval or ellipse where the variation in radius around the perimeter can be described in a simple expression or expansion of the circumferential coordinate. The current research is differentiated from the previous work in that it treats the flat and curved sections of the profile as individual structural entities. The governing equations and boundary conditions are cast in forms specific to those sections, and continuity of parameters is enforced between them. The unifying concession is that the circumferential coordinate is taken as the generic arc length, s, rather than a flat-plate-specific y or circular-arc-specific θ.

Only a few researchers have reported on the displacement and stress behavior of finite length shells with non-circular profiles (e.g., Gavelya and Sharapova, 1986; Zhu and Cheung, 1992; Hyer and McMurray, 1999). Gavelya and Sharpova seem to have performed a rigorous analysis of various cross-sections, but the terseness of the exposition and the lack of substantial numerical results make comparison difficult. Zhu and Cheung present a very novel concept of transforming the cross-section and allude to finite shell work, but again, lack of significant numerical results hampers comparison. In comparison, Hyer and McMurray do an excellent job of presenting both their solution approach and significant numerical results, however, their work is limited to elliptical sections.

The most relevant efforts in this area have been the work of Potty, Forbes, and Thomsen, as directed by Dr. Jack Vinson at the University of Delaware. These researchers examined the two-dimensional, plane-strain or generalized plane-strain analysis of similar cross-sections. The work reported herein extends these previous efforts by eventually examining a finite length shell with axial boundary conditions.

Potty (1996), and Potty and Vinson (1997), used the theorem of MPE to both develop governing differential equations and natural boundary conditions and to formulate approximate solutions. Approximate solutions were generated by assuming trial displacement functions in the form of trigonometric series (either sine or cosine). Use of a limited number of terms in the expansions ($\leqslant 15$) caused difficulty with convergence, particularly in stress results. Using advanced partial differential equation solvers (i.e., Matlab), Forbes (1999) was able to formulate the governing differential equations for the entire cross-section as a two-dimensional problem and obtain a closed-form solution. The form of the solution, though, was very complicated (involving combinations of polynomial and trigonometric functions), and extension to three dimensions was not possible. Thomsen and Vinson (2000a,b) also examined the two-dimensional problem. Their solution was formulated in terms of a set of coupled, first-order partial differential equations. Solution to the equations was by a numerical integration scheme. This approach was very rigorous and complicated, but also was not extended to three dimensions, due to this level of rigor.

In summary, the research cited in this section provides valuable comparison points for the current efforts. The most direct comparisons will be made to the two-dimensional results of Forbes. Since these results were obtained with different methods, favorable comparisons with these data will give confidence that the results are consistent and correct. There are no direct comparisons for the three-dimensional results. The current research uses finite element analyses to corroborate both the two- and three-dimensional results.

4. Formulation

For a general elastic body, the potential energy (PE) is comprised of the strain energy of the body, the work done by surface tractions, and the work done by body forces:

$$V = \int_R W \, \mathrm{d}R - \int_{S_T} T_i U_i \, \mathrm{d}S_T - \int_R F_i U_i \, \mathrm{d}R \tag{1}$$

In Eq. (1) W is the strain energy density function (to be defined); R is the volume of the body; T_i is the ith component of the surface traction; U_i is the ith component of the deformation; F_i is the ith component of a body force; and S_T is the portion of the surface of the body where the tractions are applied. The "correct" displacements will minimize the potential energy of the body (Vinson and Sierakowski, 1987).

The word "correct" is placed in quotations as, due to the great generality of Eq. (1), the displacements will be "correct" only for the phenomena that one includes. That is, the expression in Eq. (1) can be as simple or as complex as one desires, including or neglecting effects such as transverse shear, elastic coupling, moisture, temperature, etc. It is up to the researcher to select the level of rigor, and consequently the level of solution difficulty, that is required to sufficiently describe the structural problem at hand.

In practice, the variation (extremization) of the PE expression with respect to the displacements gives the static equilibrium solution. This process can be performed with either generalized displacements (e.g., $u_i = f(x,y,z)$) or with an assumed form of the displacements (e.g., $u(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ or $u(x) = A \sin x\pi/l$). Use of generalized displacements allows the finding of the governing differential equation and the "natural" boundary conditions (i.e., those that are necessary and sufficient for the problem). This is very appealing, however it becomes extremely complicated for even modest structures, and there are entire books devoted to the application of variational methods to mechanics. General solutions of this nature involve very advanced applied mathematical concepts such as Sobolev spaces, the Poincaré–Steklov operator, complex coordinate transformations, etc. (e.g., see Chudinovich and Constanda, 2000). Because of such complicated analyses, this approach remains limited to relatively simple regions (structures). While this complexity is a drawback, the generalized approach is very important beyond finding governing equations and natural boundary conditions. Such a variational approach is the foundation for the finite element method (FEM; e.g., see Reddy and Reedy, 1984; Cook et al., 1989 or Kaliakin, 2001).

Because of the above complexity, the approach chosen for this research was to use an assumed form for the displacements. The main benefit of this choice is that much of the mathematics becomes definite and concrete, and with an appropriate choice of trial displacement function, can be evaluated in a very straightforward way. The drawback to this approach is the loss of the full solid mechanics generality of the generalized displacement approach. This trade off was considered acceptable as this research was seeking a specific solution for a specific problem.

The strain energy function is formulated as:

$$U = \int_R W \, \mathrm{d}R = \int_R \left(\frac{1}{2} \sigma_{ij} \varepsilon_{ij} \right) \mathrm{d}R \tag{2}$$

where W is the strain energy density function, σ_{ij} are the stresses, ε_{ij} are the strains, and R is the volume of the body. This is an important step, as this is where one includes the relevant strains for the body in questions. For example, the following is the strain energy density function expressed in Cartesian coordinates:

$$W = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} = \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx} + \tfrac{1}{2}\sigma_{yy}\varepsilon_{yy} + \tfrac{1}{2}\sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\varepsilon_{xy} + \sigma_{xz}\varepsilon_{xz} + \sigma_{yz}\varepsilon_{yz} \tag{3}$$

Therefore, if one neglected transverse shear deformation (TSD) and transverse normal stress, then the σ_{zz}, σ_{xz}, and σ_{yz} terms would be omitted.

Next, one uses the constitutive and strain–displacement relations to restate the strain energy in terms of displacements only. This is another important step, as here one decides whether the constitutive relations are for isotropic, orthotropic or anisotropic materials, whether they are linear or non-linear, and whether they include effects such as moisture and temperature. In addition, one must decide whether the strain–displacement relations are going to be linear or non-linear to account for large displacements.

Now that the expression is stated solely in terms of displacements, one assumes a representative form for those displacements. The possibilities are endless, limited only by the ingenuity of the analyst. Some options include polynomials, trigonometrics, infinite series, etc. Combinations of forms can also be used, such as using a polynomial in one coordinate direction while using a trigonometric function in another direction. Some examples of these are:

$$\begin{aligned} w(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ w(x) &= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \\ w(x,s) &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_{ij} x^i s^j \end{aligned} \tag{4}$$

The unknown coefficients in these expressions (a_n, b_n, a_{ij}, etc.) now become the unknowns that are sought. These assumed forms might, but do not have to, satisfy the boundary conditions on the body. If they do, the problem is easier; if they do not, it adds more complexity but is still tractable. Once the representative form is determined, it is substituted into the strain energy function, the traction term(s), and the body force term(s).

Next, one performs the indicated integrations over the volume of the body. This eliminates the spatial variables and, depending on the assumed form, is usually an uncomplicated integration. After this, one takes the variation, with respect to each of the variable coefficients in the assumed form, of the final integrated PE expression. (With the use of the definite, assumed form for the displacements, this amounts to taking partial differentials of the PE expression with respect to each of the variable coefficients—the Raleigh–Ritz method.) The variation is then set equal to zero. For the current problem, this results in a set of linear algebraic equations in the unknown coefficients. The solution of this system provides the coefficients for the assumed form. Once the coefficients are known, they can be substituted back into the assumed displacement expression and any derived quantities (such as stresses) can be calculated. This is essentially a multi-variable extremization problem.

Depending on the assumed form of the displacement and the boundary conditions, there may be more variables than equations. If the assumed form satisfies all the boundary conditions, this does not occur. Usually this is only possible with simple domains and straightforward boundary conditions (i.e., clamped, simply supported). The trigonometric functions are usually found to work best in this case. However, complicated domains and/or boundary conditions make this difficult. If the assumed form does not satisfy all the boundary conditions, there are other methods for developing the set of equations. Some of these are elimination, constraint (“simple” multivariate optimization with constraints), and quadratic programming (“robust” optimization with constraints). Optimization of functions with or without equality or inequality constraints is itself a broad subject. Further very rigorous and technical information on optimization can be found in many references (e.g., Gill and Murray, 1974; Fletcher, 1981; Bertsekas, 1982).

While the elimination method (Greenberg, 1998) is appropriate for simple problems, the constraint method or more accurately, constrained minimization via Lagrange multipliers, overcomes the faults of the elimination method for more complicated problems. In the constraint method, one substitutes the assumed form for the displacements into the various boundary condition expressions (say N of these). Then, instead of solving these expressions for some of the coefficients, each boundary condition expression is treated as a constraint on the problem, is set equal to zero, and a unique Lagrange multiplier is applied to each. The assumed form for the displacements (with M unknown coefficients) is also substituted into the PE expression. This PE expression is then modified by adding in (or subtracting; it does not matter) the N boundary condition expressions with their respective Lagrange multipliers. The variation of the PE expression, with respect to the unknown coefficients is now taken (giving M equations in $M + N = P$ unknowns; the Lagrange multipliers are also treated as unknowns). After the variation is taken, the boundary condition expressions are used to obtain a “square” system (M equations $+ N$ boundary conditions $= P$ equations). This method is rigorously described in Fletcher (1981) or Greenberg (1998).

For the current work, the PE expression is being extremized. However, rather than just a few coefficients and one constraint, there will be numerous unknown coefficients and a significant number of constraints, λ_i, depending on the definition of the trial functions and boundary conditions. It will be seen subsequently that the objective function (the PE expression) is at most quadratic, and that the constraints (the boundary conditions) are linear equality constraints. These facts are beneficial simplifications to the more complex three-dimensional problem.

In the implementation of this method, the PE expression and its variation were developed analytically. After taking the variation and setting it equal to zero, the problem that remains is a linear system. To solve this, the double-precision IMSL routine DLSLSF was used (Anonymous, 1994). This routine solves a real symmetric system of linear equations (i.e., $\mathbf{Ax} = \mathbf{b}$) without iterative refinement. After the variation was found analytically, a FORTRAN program was developed to compute the necessary coefficients for the **A** matrix and **b** vector. These were passed to the DLSLSF routine, which returned a vector **x** with the solution (the trial displacement function coefficients).

5. Geometry, loads, boundary conditions, and solution

As a concept, the structure under examination in this research could be considered the fuselage of a large transport aircraft. The "rounded square" or "rounded rectangle" cross-section offers possible advantages in cargo-carrying capacity and in the structural packaging of the vehicle. It is conceivable that such an aircraft would have a pressurized fuselage for passenger or cargo comfort. If one assumes a conventional wing–body–tail arrangement of such an aircraft, then the fuselage is likely to be long and have a constant cross-section over a considerable portion of its length. We assume that the ends of the fuselage are closed in the usual manners; the front by a cockpit and cargo door, the back by the empennage and another cargo door. The loading is therefore a distributed, constant internal pressure.

As a beginning, the constant cross-section portion of the structure is examined (see Figs. 1 and 2). End effects are neglected, as is any longitudinal variation in loading, geometry, material or boundary conditions. Consequently, the analysis is reduced from a three-dimensional structure to that of a two-dimensional ring under plane strain assumptions. If the axial (plane strain) direction of the fuselage is taken as the x-axis, the plane strain assumptions state that:

$$\varepsilon_x = 0 \quad \gamma_{xz} = 0 \quad \gamma_{yx} = 0 \tag{5}$$

One could instead perform a generalized plane strain analysis where $\varepsilon_x =$ constant, however this was not pursued at this time.

The current research places as much emphasis on finding effective and efficient analytical methods as on the determination of the structural response. The goal was to start with the simpler two-dimensional problem to develop useable analytical tools, and then attempt to extend those tools to the three-dimensional problem. Thus, the shape being analyzed is the rounded-square; it is the simplest shape of this type, and allows for the maximum use of symmetry. The composition of the structure is assumed to be a composite laminate, and more specifically a composite sandwich. The effects of TSD are included; transverse normal stress and body forces are neglected.

The full square section is reduced to a one-eighth section by symmetry. The coordinate system is a right-handed x–s–z system, with the x-direction along the axis of the cylinder, the s-direction around the circumference from the top, and the z-direction outward through the thickness. The extent of the initial flat section is from $s = 0$ to S_1. The overall extent of the one-eighth section is $s = S_2$, so that the extent of the circular arc corner is $s = (S_2 - S_1)$. The radius of the corner is designated R, and the constant internal pressure is p_i. The relationship between the radius of the corner and the overall length of the one-eighth section is given by:

$$R = \frac{4(S_2 - S_1)}{\pi} \quad S_2 = \frac{R\pi}{4} + S_1 \tag{6}$$

Because the structure is composed of straight sections connected by circular arcs, the formulation of the governing equations is different in those sections. *Definitions* of strains, stresses, and PE expressions are not common across the junction between the flat and curved section at $s = S_1$. However, logically, all physical quantities must be continuous across this junction. This leads to the possibility of having at least two general approaches to the analysis. In the first approach, the structure is segregated into the curved and straight sections, and separate displacement functions are assumed for each section. Symmetry boundary conditions are imposed at the $s = 0$, $s = S_2$ ends, while matching conditions are imposed at the junction where $s = S_1$. In the second approach, the structure is separated for PE and stress–strain calculations, but the displacement functions are assumed to cover the entire structure. For this second approach, only the symmetry boundary conditions at the ends of the structure are needed. The first approach is presented in this paper.

Symmetry of the laminate is assumed so that all B_{ij}, $(\ )_{16}$, $(\ )_{26}$, and $(\ )_{45}$ couplings are neglected. The displacement field for first-order shear deformation theory is taken to be:

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\bar{\alpha}(x,y) \\ v(x,y,z) &= v_0(x,y) + z\bar{\beta}(x,y) \\ w(x,y,z) &= w(x,y) \end{aligned} \tag{7}$$

The motivating factor for the inclusion of TSD is the disparity between in-plane and out-of-plane strengths in fiber-reinforced materials. First-order theory yields a constant value of transverse shear strain through the thickness of the plate, and thus requires shear correction factors. However, first-order theory is “by far the most efficient (i.e., increased accuracy without an increase in computational effort)” (Ochoa and Reddy, 1992). There are five independent kinematic variables in three-dimensional first-order theory: displacements u_0, v_0, and w, and the rotations $\bar{\alpha}$ and $\bar{\beta}$.

In the current “ring” analysis, one assumes plane strain, that is, $\varepsilon_x = \partial(\ )/\partial x = 0$. This eliminates the consideration of u_0 and $\bar{\alpha}$ as unknowns and allows the neglect of the variation of quantities in the x-direction. Therefore, in this analysis, there are three unknowns, v_0, w, and $\bar{\beta}$.

Based on the above assumptions, the strain–displacement relations for a flat plate become (neglecting the second-order von Kármán terms; see Vinson (1999), Eqs. (2.48)–(2.50) or Ochoa and Reddy (1992), Eq. (2.5–5)):

$$\varepsilon_y = \frac{\partial v_0}{\partial y} + z\frac{\partial \bar{\beta}}{\partial y} = \varepsilon_{y_0} + z\kappa_y \quad \varepsilon_{yz} = \frac{1}{2}\left(\bar{\beta} + \frac{\partial w}{\partial y}\right) \tag{8}$$

The strain–displacement relations for a circular shell of radius R are given as (with the s-direction along the shell, noting that $\partial s = R\,\partial\theta$; see Vinson (1993), Eqs. (15.2), (15.5), and (15.8)):

$$\varepsilon_\theta = \left(\frac{\partial v_0}{\partial s} + \frac{w}{R}\right) + z\frac{\partial \bar{\beta}}{\partial s} = \varepsilon_{\theta_0} + z\kappa_\theta \quad \varepsilon_{\theta z} = \frac{1}{2}\left(\bar{\beta} + \frac{\partial w}{\partial s} - \frac{v_0}{R}\right) \tag{9}$$

It was seen in initial efforts with the MPE method that a compact representation for the trial functions is necessary to keep algebra at a manageable level. Therefore, the current approach is to use two sets of power series functions (stated in their general summation form), one for the flat and one for the curved portions. The forms of the trial displacement functions in the flat and circular arc sections (subscripts 1 and 2, respectively) are:

$$w_1(s) = \sum_{n=0}^{\infty} a_n s^n \quad w_2(s) = \sum_{n=0}^{\infty} b_n s^n$$
$$v_1(s) = \sum_{n=0}^{\infty} c_n s^n \quad v_2(s) = \sum_{n=0}^{\infty} d_n s^n \tag{10}$$
$$\bar{\beta}_1(s) = \sum_{n=0}^{\infty} f_n s^n \quad \bar{\beta}_2(s) = \sum_{n=0}^{\infty} g_n s^n$$

There are twelve boundary conditions. At $s = 0$ there are the symmetry conditions:

$$v_1(0) = 0 \quad [Q_s(0)]_1 = 0 \quad \bar{\beta}_1(0) = 0 \tag{11}$$

Matching conditions are established at $s = S_1$:

$$w_1(S_1) = w_2(S_1) \quad [N_s(S_1)]_1 = [N_s(S_1)]_2$$
$$\bar{\beta}_1(S_1) = \bar{\beta}_2(S_1) \quad [M_s(S_1)]_1 = [M_s(S_1)]_2 \tag{12}$$
$$v_1(S_1) = v_2(S_1) \quad [Q_s(S_1)]_1 = [Q_s(S_1)]_2$$

At $s = S_2$ there are the same three conditions as at $s = 0$:

$$v_2(S_2) = 0 \quad [Q_s(S_2)]_2 = 0 \quad \bar{\beta}_2(S_2) = 0 \tag{13}$$

In Eqs. (11)–(13), N, M, and Q are the integrated normal, moment, and shear stress resultants for plates, and are defined as:

$$[N] = [A]\{\varepsilon_0\} + [B]\{\kappa\}$$
$$[M] = [B]\{\varepsilon_0\} + [D]\{\kappa\}$$
$$Q_s = 2(A_{45}\varepsilon_{xz} + A_{44}\varepsilon_{sz})$$

where temperature and moisture effects have been neglected. In the above, $[A]$, $[B]$, and $[D]$ are the sub-matrices of the overall stiffness matrix that relate the mid-plane strains, $\{\varepsilon_0\}$, and the curvatures, $\{\kappa\}$, to the integrated stress resultants. Definitions of these and further discussion can be found in, e.g., Vinson and Sierakowski (1987).

Applying boundary condition #1, $v_1(0) = 0$, gives that:

$$v_1(0) = \sum_{n=0}^{\infty} c_n(0)^n = c_0 + c_1(0) + c_2(0)^2 + c_3(0)^3 + \cdots = 0$$

$$BC\#1 \Rightarrow c_0 = 0 \tag{14}$$

Neglecting coupling terms, boundary condition #2, $Q_s(0) = 0$, gives:

$$Q_s|_{s=0} = A_{44}\left(\bar{\beta}_1 + \frac{\partial w_1}{\partial s}\right)_{s=0} = 0 \tag{15}$$

From (10) and noting that power series can be manipulated term-by-term (see e.g., Greenberg, 1998, p. 179):

$$\frac{\mathrm{d}w_1(s)}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\sum_{n=0}^{\infty} a_n s^n = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}s}(a_n s^n) = \sum_{n=1}^{\infty} n a_n s^{n-1} \tag{16}$$

Thus:

$$Q_s|_{s=0} = \sum_{n=0}^{\infty} f_n s^n + \sum_{n=1}^{\infty} n a_n s^{n-1} = f_0 + f_1(0) + f_2(0)^2 + \cdots + a_1 + 2a_2(0) + 3a_3(0)^2 + \cdots = 0 \tag{17}$$

Therefore:

$$Q_s(0) = f_0 + a_1 = 0 \tag{18}$$

From boundary condition #3, $\bar{\beta}_1(0) = 0$:

$$\bar{\beta}_1(0) = \sum_{n=0}^{\infty} f_n s^n = f_0 + f_1(0) + f_2(0)^2 + \cdots = 0 \tag{19}$$

Therefore:

$$BC\#3 \Rightarrow f_0 = 0 \tag{20}$$

This result, combined with that from Eq. (18) gives that:

$$BC\#2, BC\#3 \Rightarrow a_1 = 0 \tag{21}$$

Updating the trial displacement functions (note the change in beginning index for v_1 and $\bar{\beta}_1$):

$$\begin{aligned} w_1(s) &= a_0 + \sum_{n=2}^{\infty} a_n s^n \quad w_2(s) = \sum_{n=0}^{\infty} b_n s^n \\ v_1(s) &= \sum_{n=1}^{\infty} c_n s^n \quad v_2(s) = \sum_{n=0}^{\infty} d_n s^n \\ \bar{\beta}_1(s) &= \sum_{n=1}^{\infty} f_n s^n \quad \bar{\beta}_2(s) = \sum_{n=0}^{\infty} g_n s^n \end{aligned} \tag{22}$$

Apply boundary condition #4:

$$\begin{aligned} w_1(S_1) &= w_2(S_1) \\ a_0 + \sum_{n=2}^{\infty} a_n S_1^n &= \sum_{n=0}^{\infty} b_n S_1^n \end{aligned} \tag{23}$$

The first constraint becomes:

$$\lambda_1 \left(a_0 + \sum_{n=2}^{\infty} a_n S_1^n - \sum_{n=0}^{\infty} b_n S_1^n \right) = 0 \tag{24}$$

Apply boundary condition #5:

$$\begin{aligned} \bar{\beta}_1(S_1) &= \bar{\beta}_2(S_1) \\ \sum_{n=1}^{\infty} f_n S_1^n &= \sum_{n=0}^{\infty} g_n S_1^n \end{aligned} \tag{25}$$

The second constraint becomes:

$$\lambda_2 \left(\sum_{n=1}^{\infty} f_n S_1^n - \sum_{n=0}^{\infty} g_n S_1^n \right) = 0 \tag{26}$$

The remaining boundary conditions are applied similarly to obtain the following constraints:

$$\lambda_3 \left(\sum_{n=1}^{\infty} c_n S_1^n - \sum_{n=0}^{\infty} d_n S_1^n \right) = 0 \tag{27}$$

$$\lambda_4 \left(\sum_{n=1}^{\infty} n c_n S_1^{n-1} - \sum_{n=1}^{\infty} n d_n S_1^{n-1} - \frac{1}{R} \sum_{n=0}^{\infty} b_n S_1^n \right) = 0 \tag{28}$$

$$\lambda_5\left(\sum_{n=1}^{\infty} n f_n S_1^{n-1} - \sum_{n=1}^{\infty} n g_n S_1^{n-1}\right) = 0 \tag{29}$$

$$\lambda_6\left(\sum_{n=1}^{\infty} f_n S_1^{n} + \sum_{n=2}^{\infty} n a_n S_1^{n-1} - \sum_{n=0}^{\infty} g_n S_1^{n} - \sum_{n=1}^{\infty} n b_n S_1^{n-1} + \frac{1}{R}\sum_{n=0}^{\infty} d_n S_1^{n}\right) = 0 \tag{30}$$

$$\lambda_7\left(\sum_{n=0}^{\infty} d_n S_2^{n}\right) = 0 \tag{31}$$

$$\lambda_8\left(\sum_{n=0}^{\infty} g_n S_2^{n} + \sum_{n=1}^{\infty} n b_n S_2^{n-1} - \frac{1}{R}\sum_{n=0}^{\infty} d_n S_2^{n}\right) = 0 \tag{32}$$

$$\lambda_9\left(\sum_{n=0}^{\infty} g_n S_2^{n}\right) = 0 \tag{33}$$

In the current constraint method of solution, all of the boundary conditions not already satisfied (boundary conditions #1–#3 were satisfied identically with the elimination of a_1, c_0, and f_0) are applied as constraints to the PE expression. Each of the nine constraints is multiplied by an unknown Lagrange multiplier and added to the PE expression, V, to form a modified functional, Λ, that is then extremized.

As we cannot deal with infinite limits for the sums during actual calculations, the upper limits of the power series given in Eq. (10) are taken to be m. Thus for each of w_1, v_1, and $\bar{\beta}_1$, in the flat section there will be m unknown coefficients in each series. For each of w_2, v_2, and $\bar{\beta}_2$ in the corner there will be $(m+1)$ terms. Thus the total number of unknown coefficients for all power series becomes $3m + 3(m+1)$ or $6m + 3$. Adding in the nine Lagrange multipliers, the variation of Λ will result in a system of $6m + 12$ equations in the $6m + 12$ unknowns.

Now that the constraints have been developed, the expression for the total PE is needed. Two expressions, one each for the flat and curved parts, are combined and integrated with respect to s over their respective bounds. The x-direction extent of the slice is assumed to be of unit width, and variations in the x-direction are neglected. The expression for the PE of the flat plate portion is taken from Eq. (5.70) in Vinson (1999), while the expression for the circular shell corner is taken from Eq. (4.20) in Preissner (2002). Neglecting $\partial(\ )/\partial x$ terms and any B_{ij} or $(\ )_{45}$ coupling in the structure and applying the strain–displacement relations, the total expression is therefore:

$$\begin{aligned} V = &\int_0^{S_1}\left\{\frac{A_{22}}{2}\left(\frac{dv_1}{ds}\right)^2 + \frac{D_{22}}{2}\left(\frac{d\bar{\beta}_1}{ds}\right)^2 + A_{44}\left[\frac{\bar{\beta}_1^2}{2} + \bar{\beta}_1\frac{dw_1}{ds} + \frac{1}{2}\left(\frac{dw_1}{ds}\right)^2\right]\right\}ds \\ &+ \int_{S_1}^{S_2}\left\{\frac{A_{22}}{2}\left[\left(\frac{dv_2}{ds}\right)^2 + 2\frac{w_2}{R}\frac{dv_2}{ds} + \left(\frac{w_2}{R}\right)^2\right] + \frac{D_{22}}{2}\left(\frac{d\bar{\beta}_2}{ds}\right)^2 \right. \\ &\left. + \frac{A_{44}}{2}\left[\bar{\beta}_2^2 + 2\bar{\beta}_2\frac{dw_2}{ds} - 2\bar{\beta}_2\frac{v_2}{R} + \left(\frac{dw_2}{ds}\right)^2 - 2\frac{dw_2}{ds}\frac{v_2}{R} + \left(\frac{v_2}{R}\right)^2\right]\right\}ds \\ &- \int_0^{S_1} p_i w_1\, ds - \int_{S_1}^{S_2} p_i w_2\, ds \end{aligned} \tag{34}$$

Eq. (34) is now expressed in terms of the power series and integrated with respect to s. The constraints are added in, the variation is taken, and like variations are collected. As the variation is symmetric (i.e., $\delta(ab) = (\delta a)b + a(\delta b)$), only terms on or above the diagonal are shown (the remaining terms in the matrix are filled in as needed by symmetry). Therefore:

$$
\begin{aligned}
\delta\Lambda = 0 = {} & (\lambda_1 - p_i S_1)\,\delta a_0 \\
& + \sum_{i=2}^{m}\left[A_{44}\frac{i^2 S_1^{2i-1}}{2i-1}a_i + \sum_{j>i}^{m} A_{44}\frac{ijS_1^{i+j-1}}{i+j-1}a_j + \sum_{j=1}^{m} A_{44}\frac{iS_1^{i+j}}{i+j}f_j + S_1^i\lambda_1 + iS_1^{i-1}\lambda_6 - p_i\frac{S_1^{i+1}}{i+1}\right]\delta a_i \\
& + \sum_{i=0}^{m}\left\{\left[\frac{A_{22}}{R^2}\frac{(S_2^{2i+1} - S_1^{2i+1})}{2i+1} + \left(A_{44}\frac{i^2(S_2^{2i-1} - S_1^{2i-1})}{2i-1}\right)_{i\geqslant 1}\right]b_i\right. \\
& \quad + \sum_{j>i}^{m}\left[\frac{A_{22}}{R^2}\frac{(S_2^{i+j+1} - S_1^{i+j+1})}{i+j+1} + \left(A_{44}\frac{ij(S_2^{i+j-1} - S_1^{i+j-1})}{i+j-1}\right)_{i\geqslant 1}\right]b_j \\
& \quad + \sum_{j=0}^{m}\left[\left(\frac{A_{22}}{R}\frac{j(S_2^{i+j} - S_1^{i+j})}{i+j}\right)_{j\geqslant 1} - \left(\frac{A_{44}}{R}\frac{i(S_2^{i+j} - S_1^{i+j})}{i+j}\right)_{i\geqslant 1}\right]d_j \\
& \quad + \left(\sum_{j=0}^{m} A_{44}\frac{i(S_2^{i+j} - S_1^{i+j})}{i+j}g_j\right)_{i\geqslant 1} - S_1^i\lambda_1 - \frac{S_1^i}{R}\lambda_4 - (iS_1^{i-1}\lambda_6)_{i\geqslant 1} \\
& \quad \left. + (iS_2^{i-1}\lambda_8)_{i\geqslant 1} - p_i\frac{(S_2^{i+1} - S_1^{i+1})}{i+1}\right\}\delta b_i \\
& + \sum_{i=1}^{m}\left[A_{22}\frac{i^2 S_1^{2i-1}}{2i-1}c_i + \sum_{j>i}^{m}\left(A_{22}\frac{ijS_1^{i+j-1}}{i+j-1}\right)c_j + S_1^i\lambda_3 + iS_1^{i-1}\lambda_4\right]\delta c_i \\
& + \sum_{i=0}^{m}\left\{\left[\left(A_{22}\frac{i^2(S_2^{2i-1} - S_1^{2i-1})}{2i-1}\right)_{i\geqslant 1} + \frac{A_{44}}{R^2}\frac{(S_2^{2i+1} - S_1^{2i+1})}{2i+1}\right]d_i\right. \\
& \quad + \sum_{j>i}^{m}\left[\left(A_{22}\frac{ij(S_2^{i+j-1} - S_1^{i+j-1})}{i+j-1}\right)_{i\geqslant 1} + \frac{A_{44}}{R^2}\frac{(S_2^{i+j+1} - S_1^{i+j+1})}{i+j+1}\right]d_j \\
& \quad \left. - \sum_{j=0}^{m}\frac{A_{44}}{R}\frac{(S_2^{i+j+1} - S_1^{i+j+1})}{i+j+1}g_j - S_1^i\lambda_3 - (iS_1^{i-1}\lambda_4)_{i\geqslant 1} + \frac{S_1^i}{R}\lambda_6 + S_2^i\lambda_7 - \frac{S_2^i}{R}\lambda_8\right\}\delta d_i \\
& + \sum_{i=1}^{m}\left\{\left(A_{44}\frac{S_1^{2i+1}}{2i+1} + D_{22}\frac{i^2 S_1^{2i-1}}{2i-1}\right)f_i + \sum_{j>i}^{m}\left(A_{44}\frac{S_1^{i+j+1}}{i+j+1} + D_{22}\frac{ijS_1^{i+j-1}}{i+j-1}\right)f_j\right. \\
& \quad \left. + S_1^i\lambda_2 + iS_1^{i-1}\lambda_5 + S_1^i\lambda_6\right\}\delta f_i + \sum_{i=0}^{m}\left\{\left[A_{44}\frac{(S_2^{2i+1} - S_1^{2i+1})}{2i+1} + \left(D_{22}\frac{i^2(S_2^{2i-1} - S_1^{2i-1})}{2i-1}\right)_{i\geqslant 1}\right]g_i\right. \\
& \quad + \sum_{j>i}^{m}\left[A_{44}\frac{(S_2^{i+j+1} - S_1^{i+j+1})}{i+j+1} + \left(D_{22}\frac{ij(S_2^{i+j-1} - S_1^{i+j-1})}{i+j-1}\right)_{i\geqslant 1}\right]g_j - S_1^i\lambda_2 \\
& \quad \left. - (iS_1^{i-1}\lambda_5)_{i\geqslant 1} - S_1^i\lambda_6 + S_2^i\lambda_8 + S_2^i\lambda_9\right\}\delta g_i
\end{aligned}
\tag{35}
$$

Writing the variation of the PE expression in the general "summation" form above allows for a variable number of terms to be taken in the solution. This allows the convergence of the method to be studied. To accomplish this efficiently, and anticipating that a large number of terms might be necessary, a FORTRAN computer program was written to calculate the coefficients and solve the system.

The variation of the PE expression, as given in Eq. (35) above, is set equal to zero to determine the unknown coefficients. As the variations of the unknown coefficients (i.e., δa_i, δb_i, etc.) are by assumption non-zero, the expression within the brackets that each variation multiplies must therefore be equal to zero. This process leads to the solution of a system of the form $\mathbf{Ax} = \mathbf{b}$. Eq. (35) defines the entries to fill the coefficient matrix, $\mathbf{A}$, and load vector, $\mathbf{b}$. The coefficients of the power series are numbered consecutively from the a_is through the g_is, and these numbers define their row/column location in the matrices, and there are a total of $(6m + 3)$ coefficients. The row location is determined by which variable the variation is for, while the column location is determined by the variable that appears inside the variation brackets. For computation, the primary controlling index of the expression is i, and both i and (secondarily, when necessary) j go to the limit of m.

6. Convergence of power series solution

The results for this approach, with representation of the deflections by arbitrary-ordered power series, were compact, efficient, and accurate in two dimensions. Stress and displacement calculations for limiting cases of flat plates and circular cylinders showed appropriate and accurate behavior. The convergence behavior of the power series representation is far better than that for a trigonometric series representation, with only a modest change in convergence performance with differentiation. No more than a fourth-order representation was necessary to obtain a converged deflection, versus 500–1000 terms for a trigonometric approach (Preissner, 2002).

The convergence was so rapid and accurate that the authors wondered if the solutions for the higher-ordered polynomials were indeed different from each other. Fig. 3 shows that they are indeed different. This plot shows the lateral displacement function for the circular arc section, w_2 plotted over the entire range of

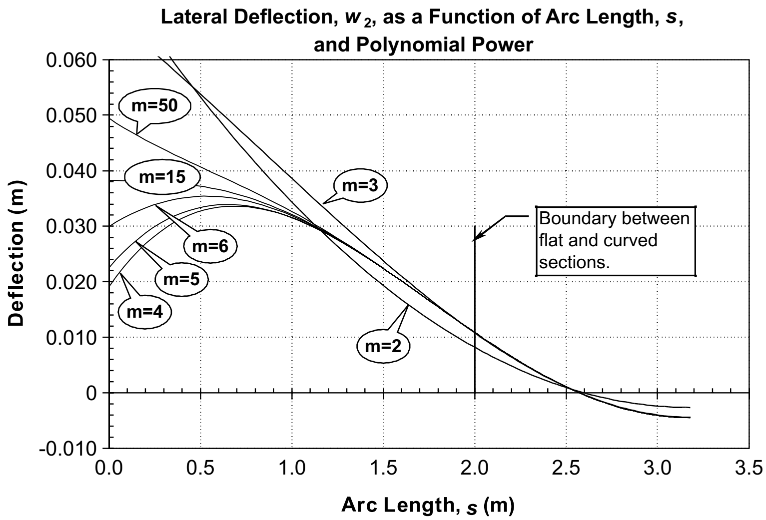

Fig. 3. Convergence of lateral deflection in the circular arc section, w_2 as a function of number of terms, m, in the power series.

arc length. Note that this solution is only meant to apply in the range $S_1 \leqslant s \leqslant S_2$. However, examining the solution in the range $0 \leqslant s \leqslant S_1$ shows that the different orders of solution do have different values in that range, despite the excellent agreement over the applicable range.

The convergence of the derivative quantities N, M, Q, upper and lower skin hoop stresses and core shear stress (not shown) were also excellent. Quantities such as N, M, Q, and σ_z, which depend on the derivative of only one primary variable converge only slightly slower than the primary variables themselves, at three to five terms. The skin stresses depend on the derivatives of two primary variables, and so converge the slowest of all; to remove nearly all of the fluctuations, 10–15 terms are needed. Even this is far better performance than the trigonometric series. Arfken (1966) notes how valuable and powerful power series are, due to their excellent convergence and ability to robustly withstand differentiation and integration as many times as necessary.

7. Results and discussion

A standard problem was used as a basis for comparison of the different methodologies (Preissner, Forbes, and Thomsen) being developed. The geometry of this standard was loosely based on the size of the cargo compartment of a Lockheed C-5A Galaxy transport aircraft. The geometry for the current research was limited to a "rounded square" cross-section, due to the formulation of the MPE solution methods. The reader is referred to Fig. 1(a) and (b) for diagrams of the full cross-section and the portion that was analyzed.

Geometry and loading:

$S_1 = 2.0$ m	Length of the horizontal flat section from the vertical midplane to the beginning of the circular arc corner section.
$R = 1.5$ m	Radius of the corner section.
$t_1 = 0.005$ m	Thickness of the inner (lower) face of the sandwich.
$t_3 = 0.005$ m	Thickness of the outer (upper) face of the sandwich.
$h_c = 0.20$ m	Thickness of the sandwich core.
$p_i = 0.1$ MPa	Internal pressure.

Materials

Inner face	T300/5208 unidirectional hoop wrap (all fibers in the hoop, i.e., circumferential, direction).
Outer face	T300/5208 unidirectional hoop wrap.
Core	Klegecell foam.

Material properties

T300/5208 carbon/epoxy

$E_{11} = 153.0$ GPa $\quad E_{22} = 10.9$ GPa $\quad G_{12} = 5.6$ GPa $\quad \nu_{12} = \nu_{13} = 0.3 \quad \nu_{23} = 0.02137$

Klegecell foam (Anonymous, 2000)

$E_{11} = E_{22} = 145.7$ MPa $\quad E_{33} = 182.7$ MPa $\quad G_{13} = G_{23} = 69.0$ MPa $\quad G_{12} = 55.2$ MPa

$\nu_{12} = \nu_{13} = 0.3$

Because of its comprehensive and well-documented nature, the finite element results of ABAQUS are taken as the "best" answer. The ABAQUS analysis used the S4R shell element (Anonymous, 1997), with a grid of eight elements in the flat section and eight elements in the corner. No significant discrepancies between the MPE and ABAQUS results were apparent. In hindsight, this agreement should be anticipated as both the FEM and MPE methods are formulated based on variational principles. The agreement between the FEM/MPE methods and the Forbes results is encouraging due to the disparate methods used to obtain said results. As the Forbes results also matched those of Thomsen, only the Forbes results are shown for clarity.

Fig. 4 compares the lateral deflection results, and it is seen that Forbes and the MPE results underpredict the maximum deflection by approximately 2.5% and 2.8% and underpredict the minimum deflection by 0.1% and 4.3%, respectively. Fig. 5 compares the in-plane deflection results; the MPE data is seen to be very

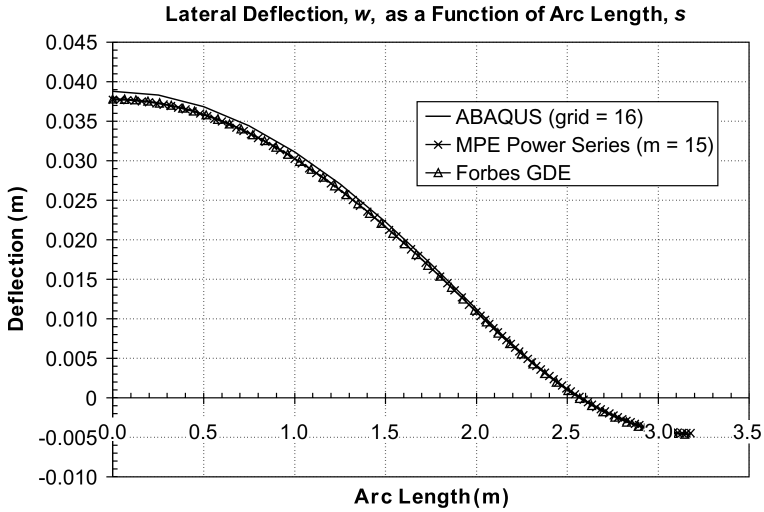

Fig. 4. Comparison of lateral deflection, w from ABAQUS, current MPE, and Forbes GDE methods.

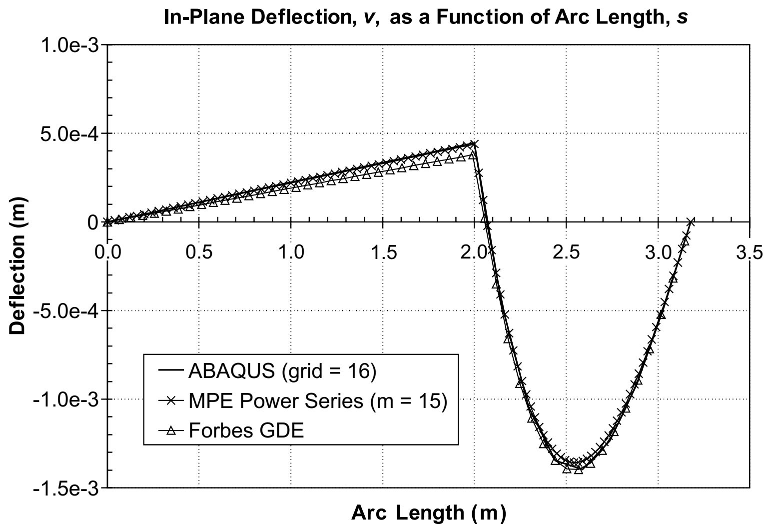

Fig. 5. Comparison of in-plane deflection, v from ABAQUS, current MPE, and Forbes GDE methods.

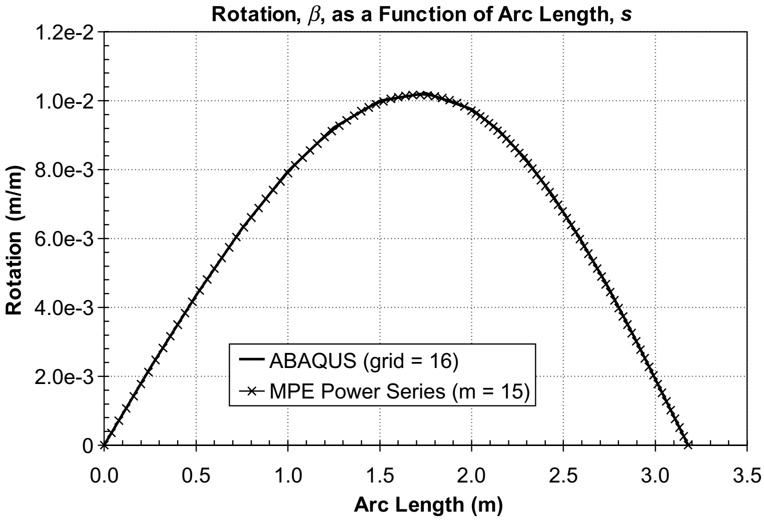

Fig. 6. Comparison of normal rotation, $\bar{\beta}$ from ABAQUS and current MPE methods.

close to the ABAQUS results in the straight section and underpredict the deflection in the curved section, while the Forbes data has the opposite behavior. Fig. 6 compares the normal rotation, $\bar{\beta}_s$, between ABAQUS and the MPE method; the MPE results are in very close agreement. Data on the rotation was not available for the Forbes method.

Comparison of skin hoop stress data is shown in Figs. 7 and 8; the agreement between the three methods is excellent. Comparison of core shear stress is shown in Fig. 9; agreement here is not as good as with the skin stress. At the ends, agreement is good, but both MPE and Forbes under predict the shear stress near the junction by about 8%. In the MPE formulation, the core shear stress is a function of the core shear modulus, G the normal rotation, $\bar{\beta}_s$, and the slope of the lateral deflection, $\partial w/\partial s$. As G is an input to

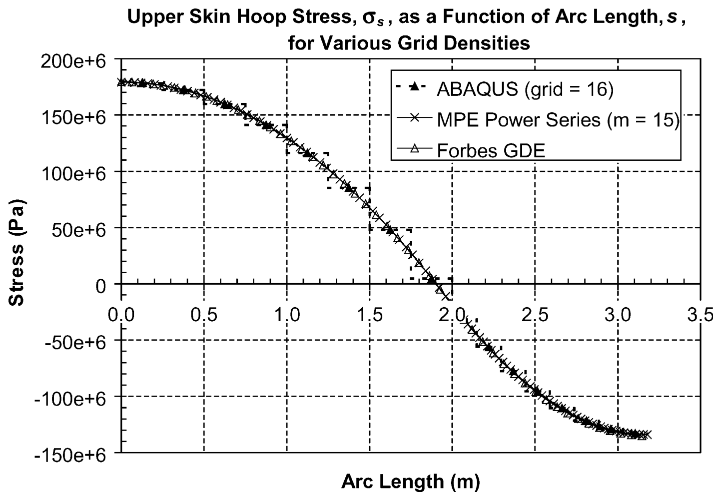

Fig. 7. Comparison of upper skin hoop stress, σ_s from ABAQUS, current MPE, and Forbes GDE methods.

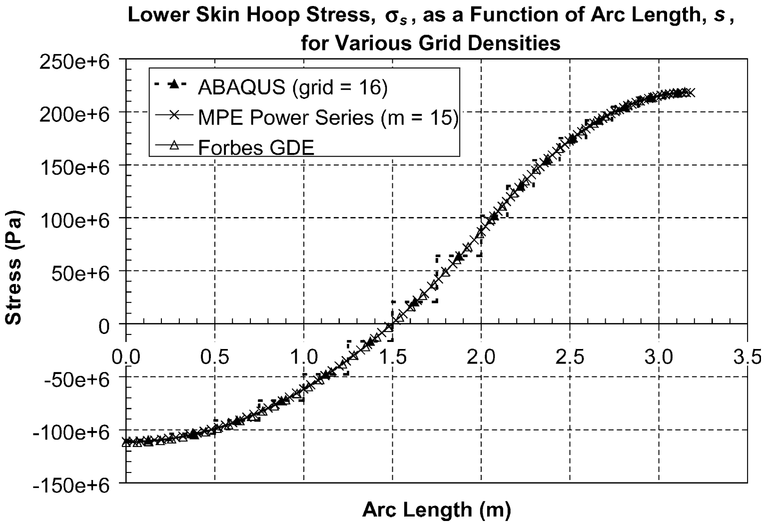

Fig. 8. Comparison of lower skin hoop stress, σ_s from ABAQUS, current MPE, and Forbes GDE methods.

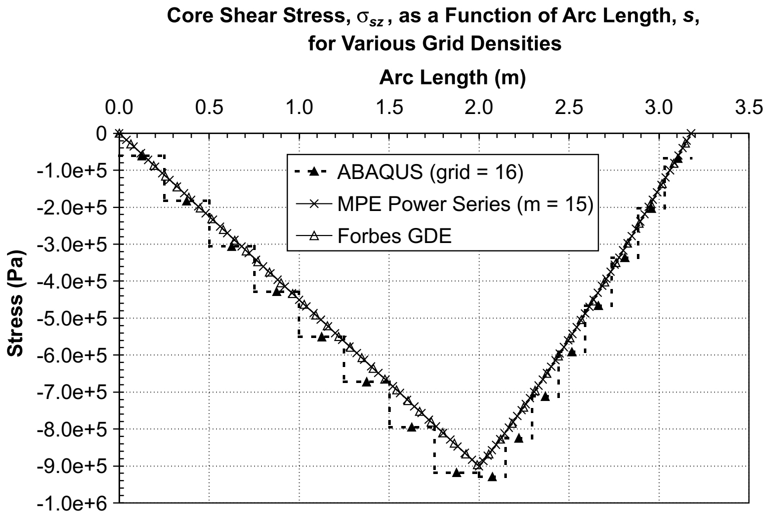

Fig. 9. Comparison of core shear stress, σ_{sz} from ABAQUS, current MPE, and Forbes GDE methods.

ABAQUS, this should not be a factor, and it is seen in Fig. 6 that the rotations agree. Fig. 10 shows that the slope of the lateral deflection for the MPE and Forbes solutions are about 3% less than that calculated from the ABAQUS deflections, which contributes to the difference in core stress. The balance of the difference is likely due to differences in formulation for the calculation of the stress between MPE and ABAQUS. All MPE calculations shown in Figs. 4–10 are for 15 terms in the series, i.e., $m = 15$.

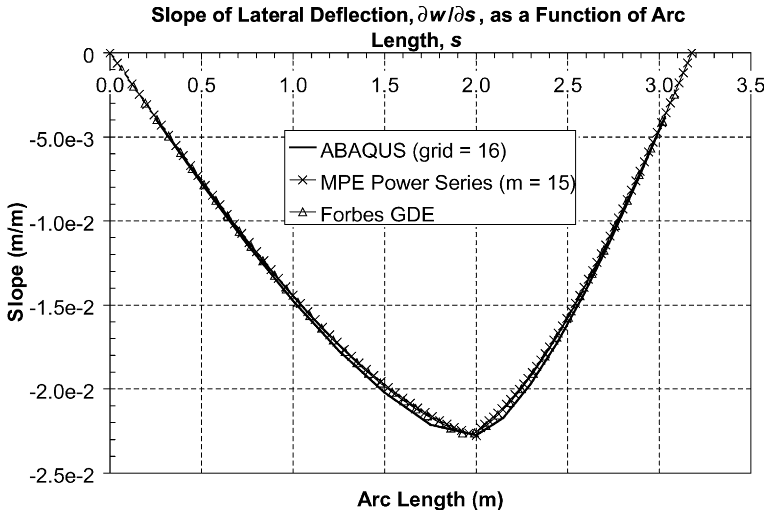

Fig. 10. Comparison of slope of lateral deflection, $\partial w/\partial s$ from ABAQUS, current MPE, and Forbes GDE methods.

8. Conclusions

Extension of this solution methodology to other shapes, such as a "rounded rectangle", a trapezoid, an ogive, etc., should be straightforward. Any shape that could be broken into flat sections, circular arc sections, elliptical sections, tapered sections, etc. could be handled by formulating the PE expression for each of these basic shapes, and then assembling the expressions appropriately. Inclusion of the desired physical phenomena, such as B_{ij} coupling and even non-linear effects, could be incorporated. Indeed, once the PE expressions for the subsections were developed, it would be feasible to combine them into one larger computer program that would allow many sections to be readily joined together in different ways.

The results of the MPE analysis have excellent accuracy as compared to those of a finite element analysis. This work shows how the restriction of meeting boundary conditions a priori with the displacement trial functions can be lifted to allow for a wider choice of functions. Although there is a lack of experimental data for comparison, the good agreement between the MPE method, the Forbes closed-form solution, the Thomson integral solution, and the finite element analysis increases the confidence that all methods are predicting the structural response properly. The MPE method provides an alternative to the two-dimensional closed-form solution from Matlab (Forbes, 1999), as this could not be extended to three dimensions. The MPE method had a short solution time, and was flexible within the parameters for which it was programmed. However, as formulated, it was not as flexible as the other methods, and would need to be improved as described in the previous paragraph. In addition, although it had been hoped in the beginning that the MPE method would be simpler to formulate than the other methods, in the end, the initial set-up and learning curve were just as complicated. However, any future work will benefit from this investment of effort.

Acknowledgements

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References

Anonymous, 1994. IMSL MATH/LIBRARY User's Manual, Version 3.0. Visual Numerics, Inc., Houston, Texas.

Anonymous, 1997. ABAQUS/Standard User's Manual, Vols. I–III. Hibbitt, Karlsson, and Sorensen, Inc., Pawtucket, Rhode Island.

Anonymous, 2000. Klegecell R Technical Data Sheet. DIAB Technologies, DeSoto, Texas.

Arfken, G.G., 1966. Mathematical Methods for Physicists. Academic Press, New York.

Bergman, R.M., 1973. Investigation of the free vibrations of noncircular cylindrical shells, English translation of Prikladnaya Matematika i Mekhanika. Journal of Applied Mathematics and Mechanics 37 (6), 1068–1077.

Bertsekas, D.P., 1982. Constrained Optimization and Lagrange Multiplier Methods. Academic Press, New York.

Chroscielewski, J., Makowski, J., Stumpf, H., 1997. Finite element analysis of smooth, folded and multi-shell structures. Computer Methods in Applied Mechanics and Engineering 141 (1–2), 1–46.

Chudinovich, I., Constanda, C., 2000. Variational and potential methods in the theory of bending of plates with transverse shear deformation. In: Monographs and Surveys in Pure and Applied Mathematics, vol. 115. CRC Press, Boca Raton, FL.

Cook, R.D., Malkus, D.S., Plesha, M.E., 1989. Concepts and Applications of Finite Element Analysis, third ed. John Wiley and Sons, New York.

Crull, M.M., Basu, P.K., 1994. hp version of finite-element mixed method. Journal of Engineering Mechanics 120 (11), 2342–2360.

Dym, C.L., 1974. Introduction to the Theory of Shells, first ed. Pergamon Press, Glasgow.

Fletcher, R., 1981. Practical methods of optimization. In: Constrained Optimization, vol. 2. John Wiley and Sons, Chichester.

Flügge, W., 1943. Statik und Dynamik der Schalen. Edwards Brothers, Ann Arbor.

Forbes, B., 1999. A Thesis on solution methods for 2-D rings with asymmetry. Master's Thesis, University of Delaware, Newark.

Gavelya, S.P., Sharapova, D.I., 1986. Deformation of noncircular cylindrical shells, English Translation of Prikladnaya Mekhanika. Soviet Applied Mechanics 22 (1), 52–56.

Gill, P.E., Murray, W., 1974. In: Gill, P.E., Murray, W. (Eds.), Numerical Methods for Constrained Optimization. Academic Press, London.

Greenberg, M.D., 1998. Advanced Engineering Mathematics, second ed. Prentice Hall, New Jersey.

He, J.F., 1992. Cylindrical bending of arbitrary cross-ply laminated plates. Composite Structures 21 (2).

Hyer, M.W., McMurray, J.M., 1999. Internally pressurized elliptical composite cylinders. Composite Structures 46 (1), 17–31.

Kaliakin, V.N., 2001. Approximate Solution Techniques, Numerical Modeling and Finite Element Methods. Marcel Dekker, Inc., New York.

Kapania, R.K., 1989. Review on the analysis of laminated shells. Journal of Pressure Vessel Technology Transactions of the ASME 111 (2), 88–96.

Kumar, V., Singh, A.V., 1993. Vibration analysis of non-circular cylindrical shells using Bezier functions. Journal of Sound and Vibration 161 (2), 333–354.

Kumar, V., Singh, A.V., 1997. Geometrically non-linear dynamic analysis of laminated shells using Bezier functions. International Journal of Non-Linear Mechanics 32 (3), 425–442.

Lim, C.W., Liew, K.M., Kitipornchai, S., 1998. Vibration of cantilevered laminated composite shallow conical shells. International Journal of Solids and Structures 35 (15), 1695–1707.

Love, A.E.H., 1888. The small free vibrations and deformation of a thin elastic shell. Philosophical Transactions of the Royal Society of London 17, 491–546.

McDaniel, J.G., Ginsberg, J.H., 1993. Thickness expansions for higher-order effects in vibrating cylindrical shells. Journal of Applied Mechanics 60 (2), 463–469.

Messina, A., Soldatos, K.P., 1999. Vibration of completely free composite plates and cylindrical shell panels by a higher-order theory. International Journal of Mechanical Sciences 41 (8), 891–918.

Naghdi, P.M., 1956. A survey of recent progress in the theory of elastic shells. Applied Mechanics Reviews 9 (9), 365–368.

Ochoa, O.O., Reddy, J.N., 1992. Finite Element Analysis of Composite Laminates. Kluwer Academic Publishers, Dordrecht, The Netherlands.

Potty, K.P.K., 1996. Use of Composite and Sandwich Materials in Complex Shell Structures. Ph.D. Dissertation, University of Delaware, Newark.

Potty, K.P.K., Vinson, J.R., 1997. Behavior of composite material sandwich ring and shell in complex structures. In: Allen, H.G. (Ed.), Sandwich construction 3—Proceedings of the 3rd International Conference on Sandwich Construction. EMAS Publishers, London, pp. 281–292.

Preissner, E.C., 2002. Analysis of Cylindrical Composite Shells: Examining the Effects of Asymmetric Construction and a Rounded-square Cross-section. Ph.D. Dissertation, University of Delaware, Newark.

Reddy, J.N., Reedy, J.N., 1984. Energy and Variational Methods in Applied Mechanics. John Wiley, New York.

Simitses, G.J., 1986. Buckling and postbuckling of imperfect cylindrical shells: a review. Applied Mechanics Reviews 39 (10), 1517–1524.

Simitses, G.J., 1996. Buckling of moderately thick laminated cylindrical shells: a review. Composites Part B: Engineering 27 (6), 581–587.

Singer, J., 1982. Buckling experiments on shells—A review of recent developments. Solid Mechanics Archives 7 (3).

Sklepus, S.N., 1996. Thermoelasticity of laminated shallow shells of complex form. International Applied Mechanics 32 (4), 281–285.

Soldatos, K.P., Messina, A., 1998. Vibration studies of cross-ply laminated shear deformable circular cylinders on the basis of orthogonal polynomials. Journal of Sound and Vibration 218 (2), 219–243.

Soldatos, K.P., 1984. A Flügge-type theory for the analysis of anisotropic laminated non-circular cylindrical shells. International Journal of Solids and Structures 20 (2), 107–120.

Suzuki, K., Leissa, A.W., 1990. Analysis of free vibrations of noncircular thick cylindrical shells having circumferentially varying thickness. International Journal of Solids and Structures 26 (3), 255–270.

Suzuki, K., Shikanai, G., Leissa, A.W., 1996. Free vibrations of laminated composite non-circular thick cylindrical shells. International Journal of Solids and Structures 33 (27), 4079–4100.

Suzuki, K., Tamura, S., Kosawada, T., Takahashi, S., 1983. Vibrations of noncircular cylindrical shells. Bulletin of the JSME 26 (215), 818–826.

Tamurov, Y.N., 1990. Modification of a generalized theory of shallow sandwich shells taking into account squeezing of the physically nonlinear filler, English Translation of Prikladnaya Mekhanika. Soviet Applied Mechanics 26 (12), 1149–1154.

Thomsen, O.T., Vinson, J.R., 2000a. Design study of non-circular pressurized sandwich fuselage section using a high-order sandwich theory formulation. Proceedings of the Fifth International Conference on Sandwich Construction, 2000.

Thomsen, O.T., Vinson, J.R., 2000b. Comparative study of two different conceptual design principles for non-circular pressurized sandwich fuselage sections using a high-order sandwich theory formulation. ASME AD-vol. 62/AMD-vol. 245, Mechanics of Sandwich Structures, 2000. ASME, pp. 151–168.

Timoshenko, S., Woinowsky-Krieger, S., 1959. Theory of Plates and Shells, second ed. McGraw-Hill, New York.

Vinson, J.R., Sierakowski, R.L., 1987. The Behavior of Structures Composed of Composite Materials. Kluwer Academic Publishers, Dordrecht.

Vinson, J.R., 1993. The Behavior of Shells Composed of Isotropic and Composite Materials. Kluwer Academic Publishers, Dordrecht.

Vinson, J.R., 1999. The Behavior of Sandwich Structures of Isotropic and Composite Materials. Technomic Publishing, Lancaster.

Wenda, L., 1985. Doubly curved shallow shell with rectangular base elastically supported by edge arched beams and tie-rods. International Association for Shell and Spatial Structures International Congress: Theory and Experimental Investigation of Spatial Structures, Application of Shells in Engineering Structures 1, 350–367.

Wu, C.-P., Liu, C.-C., 1994. Stress and displacement of thick doubly curved laminated shells. Journal of Engineering Mechanics 120 (7), 1403–1428.

Wu, C.-P., Tseng, K.-W., Kuo, L.-C., 1995. Vibration and stability of simply supported cross-ply doubly curved shells. Journal of the Chinese Society of Mechanical Engineers 16 (3), 221–237.

Xiaoyu, J., 1997. 3-D vibration analysis of fiber reinforced composite laminated cylindrical shells. Journal of Vibration and Acoustics, Transactions of the ASME 119 (1), 46–51.

Ying-guang, F., Ji-hao, P., Wei-xin, C., 1992. Theory of thick-walled shells and its application in cylindrical shell. Applied Mathematics and Mechanics (English ed.) 13 (11), 1055–1065.

Zhu, J.X., Cheung, Y.K., 1992. Analysis of closed cross-section thin-walled structures. Thin-Walled Structures 14 (1), 21–35.